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Algorithm of Price Adjustment for Market Equilibrium

Yurii Nesterov and Vladimir Shikhman

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DISCUSSION PAPER

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Yurii NESTEROV* and Vladimir SHIKHMAN†

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Abstract

In this paper, we suggest an algorithm for price adjustment towards a partial market equilibrium. Its convergence properties are crucially based on Convex Analysis. Our price adjustment corresponds to a subgradient scheme for minimizing a special nonsmooth convex function. This function is the total excessive revenue of the market's participants [16, 18], and its minimizers are equilibrium prices. As the main result, the algorithm of price adjustment is shown to converge to equilibrium prices. Additionally, a market equilibrium clears on average during the price adjustment process. This means that the market clears by historical averages of supply and demand. Moreover, an efficient rate of convergence is obtained. Additionally, we endow our algorithm with decentralized prices by introducing the trade design. The latter suggests that producers settle and update their individual prices, and consumers buy at the lowest purchase price. The proposed price adjustment enjoys a natural behavioral interpretation. First, producers forecast their individual prices to be proportional to their excess demands. For the price update, they subsequently apply an average of these price forecasts over time.

Keywords: price adjustment, nonsmooth convex optimization, subgradient methods, decentralization of prices, partial equilibrium, historical averaging

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CORE DISCUSSION PAPER

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Algorithm of price adjustment for market equilibrium

Yu. Nesterov [*] V. Shikhman [†]

January 25, 2015

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1 Introduction

In microeconomics, partial equilibrium describes a state of a market where supply of goods meets demand by an appropriate price. This market clearing or competitive price is often referred to as equilibrium price. Naturally, the question arises on how the prices can be adjusted in order to reach an equilibrium. In order to address this question, we study the following price adjustment process:

> **Phase I, Production.** Let producers face their individual prices of goods. They maximize profits by computing the optimal production bundles. Additionally, producers compare their profits with fixed costs and decide whether to produce or not.
>
> **Phase II, Consumption.** Consumers identify the lowest purchase prices of goods, and minimize expenditures by computing the optimal consumption bundles. Then, they compare their expenditures with wealths and decide whether to consume or not. Finally, consumers decide on demands from those producers who offer lowest purchase prices.
>
> **Phase III, Price update.** Producers face their individual excess supplies, and accumulate them over time. They set the price forecast being proportional to their accumulated excess supplies. In order to obtain new individual prices, producers combine their previous prices with forecasts.

In this paper, we show that the above algorithm of price adjustment successively leads to a market equilibrium. Its convergence properties are crucially based on Convex Analysis. The price adjustment corresponds to the quasi-monotone subgradient method for nonsmooth convex minimization, recently suggested in [17]. As objective function for the latter method we take the total excessive revenue of the market. Equilibrium prices can be then characterized as its minimizers. Overall, we claim that our price adjustment is reliable, computable, and decentralized.

Reliability refers to the fact that the price adjustment converges to a market equilibrium *on average*. First, note that the prices in Phase III are formed by averaging of price forecasts. This behavioral pattern accommodates the producers' *experience* during the price evolvement. Secondly, the sequence of successful and failed production and consumption attempts in Phases I and II defines the agents' individual history. On average, this individual history provides production and consumption frequencies, which approach the *participation levels* of producers and consumers in market activities over time [16, 18]. Thirdly, average production and consumption bundles implemented in Phases I and II are shown to approximately clear the market. The latter means that during the price adjustment supply meets demand *statistically*. In mathematical terms, frequencies and average bundles approach the solution of the adjoint problem for the minimization of the total excessive revenue.

Computability of price adjustment means that we can guarantee its rate of convergence. In worst case, the number of price updates needed to achieve the ε-tolerance is proportional to $\frac{1}{\varepsilon^2}$. Note that this rate of convergence is optimal for nonsmooth convex minimization, cf. [15]. From the economic perspective, this result explains why competitive markets adjust in efficient way, moreover, it quantifies the worst-case efficiency.

Decentralization explains how market participants can successively update prices by themselves rather than by relying on a central authority. In this article, the decentralization of prices is implemented by the *trade design*:

producers suggest and update their individual *selling prices,*
and consumers buy at the lowest *prices observed at the market.*

It is crucial for our approach that the introduction of the trade design preserves convexity of the total excessive revenue. Moreover, its convex subgradients with respect to producer's prices become the *individual* excess supplies, which are easily observable. This advantage is used by producers in Phase III for computing the individual price forecasts.

We mention some literature related to our approach. The study of convex market models has been initiated in [5]. Based on that, polynomial-time algorithms for equilibrium pricing are studied in [1, 2, 3]. Recently, Convex Analysis was used to model financial equilibria, see [9]. Decentralization of prices via the trade design is suggested already in [4]. There a disequilibrium price dynamics due to Hahn is considered, see [7]. We refer to [8] for an overview on prices adjustment processes. We also mention [13] where a simultaneous ascending auction is used to construct a decentralized price adjustment. Starting with [21], the theory of bargaining has been successively applied for market decentralization (see e.g. [6, 14]).

The article is organized as follows. In Section 2 we present the excessive revenue model and introduce our concept of market equilibrium. In Section 3 we describe the decentralization of prices. We prove convergence of the decentralized price adjustment in Section 4. Appendix is devoted to the mathematical justification of quasi-monotone subgradient schemes.

Notation. Our notation is quite standard. We denote by $\mathbb{R}^n$ the space of n-dimensional column vectors $x = (x^{(1)}, \ldots, x^{(n)})^T$, and by $\mathbb{R}^n_+$ the set of all vectors with nonnegative components. For x and y from $\mathbb{R}^n$, we introduce the standard scalar product and the Hadamard product

$$\langle x, y \rangle = \sum_{i=1}^{n} x^{(i)} y^{(i)}, \quad x \circ y = \left(x^{(i)} y^{(i)} \right)_{i=1}^{n} \in \mathbb{R}^n.$$

Finally, $(a)_+$ denotes the positive part of the real value $a \in \mathbb{R}$: $(a)_+ = \max\{a, 0\}$. For $x = (x^{(1)}, \ldots, x^{(n)})^T \in \mathbb{R}^n$ we denote $(x)_+ = \left((x^{(1)})_+, \ldots, (x^{(n)})_+\right)^T$.

2 Excessive revenue model

We present the *excessive revenue model* of a competitive market with multiple goods, which has been introduced in [16] and extended in [18]. For that, we need to describe the behavior of producers and consumers. Given prices of goods, they maximize their profits and minimize their expenditures by deciding on tentative production and consumption patterns. The latter must be compatible with their needs and technological constraints. Additionally, producers compare their profits with fixed costs, and consumers compare their expenditures with available wealths. This comparison is needed to adjust real production and consumption bundles accordingly. Then, we define the equilibrium production and consumption flows, which clear the market of goods by some equilibrium prices.

2.1 Producers and consumers

Consider a market with K producers, which are able to produce n different goods. Given a vector of prices $p \in \mathbb{R}^n_+$, the k-th producer forms his supply operator $S_k(p)$ of real production bundles $\widetilde{y}_k \in \mathbb{R}^n_+$. For that, the k-th producer maximizes the profit with respect to his variable cost, subsequently he tries to cover his fixed cost. Namely,

- The k-th producer chooses first the *tentative production* bundle $y_k \in \mathbb{R}^n_+$ by solving the *profit maximization* problem:

$$\pi_k(p) \stackrel{\mathrm{def}}{=} \max_{y_k \in \mathcal{Y}_k} \langle p, y_k \rangle - c_k(y_k). \tag{1}$$

  Here, $\mathcal{Y}_k \subset \mathbb{R}^n_+$ is the production set, assumed to be nonempty, compact and convex. The producer's yield is $\langle p, y_k \rangle$. The variable cost of producing y_k is denoted by $c_k(y_k)$. We assume that c_k is a convex function on $\mathbb{R}^n_+$. Clearly, the profit $\pi_k(p)$ is convex in p as the maximum of linear functions. By $\mathcal{Y}^*_k(p)$ we denote the set of optimal solutions of (1), i.e. $y_k \in \mathcal{Y}^*_k(p)$. Note that the profit maximization problem (1) appears already in Marshallian partial equilibrium analysis (see e.g. [12]).

- Secondly, the k-th producer compares his profit $\pi_k(p)$ with his fixed cost of maintaining the technological set $\mathcal{Y}_k$, denoted by $\kappa_k \equiv \kappa_k(\mathcal{Y}_k) \in \mathbb{R}_+$. The latter can include the interest paid to the bank, different charges for renting the equipment, land use, etc. By this comparison, a participation level $\alpha_k \equiv \alpha_k(p) \in [0, 1]$ of k-th producer is *properly* adjusted:

$$\alpha_k(p) \stackrel{\mathrm{def}}{=} \begin{cases} 1, & \text{if } \pi_k(p) > \kappa_k, \\ 0, & \text{if } \pi_k(p) < \kappa_k. \end{cases} \tag{2}$$

- Finally, the *supply* operator $S_k : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n_+$ of the k-th producer is given by

$$S_k(p) \stackrel{\mathrm{def}}{=} \left\{ \widetilde{y}_k = \alpha_k y_k \,|\, \alpha_k \equiv \alpha_k(p) \text{ and } y_k \in \mathcal{Y}^*_k(p) \right\}. \tag{3}$$

  Here, the *real production* bundles are

$$\widetilde{y}_k \stackrel{\mathrm{def}}{=} \alpha_k y_k,$$

  where $\alpha_k \equiv \alpha_k(p)$ is a proper participation level of the k-th producer, and $y_k \in \mathcal{Y}^*_k(p)$ is his tentative production.

Let I consumers be active at the market. The i-th consumer has to decide on his real consumption bundle $\widetilde{x}_i \in \mathbb{R}^n_+$. These real consumption bundles form his demand $D_i(p)$, given the price $p \in \mathbb{R}^n_+$. The i-th consumer minimizes the expenditure with an aim to guarantee his desirable utility level. Then he tries to cover this expenditure by the available wealth. Namely,

- The i-th producer decides first on the *tentative consumption* bundle $x_i \in \mathbb{R}^n_+$ by minimizing his expenditure:

$$e_i(p) \stackrel{\mathrm{def}}{=} \min_{\substack{x_i \in X_i \\ u_i(x_i) \geq u_i}} \langle p, x_i \rangle = \min_{x_i \in \mathcal{X}_i} \langle p, x_i \rangle, \tag{4}$$

where the i-th consumption set is

$$\mathcal{X}_i \stackrel{\mathrm{def}}{=} \{x_i \in X_i \,|\, u_i(x_i) \geq u_i\}.$$

Here, $X_i \subset \mathbb{R}^n_+$ is assumed to be nonempty, compact and convex. By $u_i : X_i \to \mathbb{R}_+$ we denote the utility function of the i-th consumer, assumed to be concave. The utility level $u_i \in \mathbb{R}_+$ is desirable by i-th consumer. The consumer's expenditure $e_i(p)$ is concave in p as the minimum of linear functions. By $\mathcal{X}^*_i(p)$ we denote the set of optimal solutions of (4), i.e. $x_i \in \mathcal{X}^*_i(p)$. The minimization of expenditure in (4) is well-known in economics as a dual problem for utility maximization. The desirable utility level u_i mainly reflects the consumer's standards on qualities of goods. In [20] the agent who faces the expenditure minimization problem (4) is called the dual consumer. We also refer to [10, Chapter 10] and [11] for more details on the dual theory of consumption.

- Secondly, the i-th consumer compares his expenditure $e_i(p)$ with his available wealth $w_i \in \mathbb{R}_+$. The latter can include the budget, salary and rent payments, etc. By this comparison, a participation level $\beta_i \equiv \beta_i(p) \in [0, 1]$ of i-th consumer is *properly* adjusted:

$$\beta_i(p) \stackrel{\mathrm{def}}{=} \begin{cases} 1, & \text{if } e_i(p) < w_i, \\ 0, & \text{if } e_i(p) > w_i. \end{cases} \tag{5}$$

- Finally, the *demand* operator $D_i : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n_+$ of the i-th consumer is given by

$$D_i(p) \stackrel{\mathrm{def}}{=} \{\widetilde{x}_i = \beta_i x_i \,|\, \beta_i \equiv \beta_i(p) \text{ and } x_i \in \mathcal{X}^*_i(p)\}. \tag{6}$$

Here, the *real consumption* bundle are

$$\widetilde{x}_i \stackrel{\mathrm{def}}{=} \beta_i x_i,$$

where $\beta_i \equiv \beta_i(p)$ is a proper participation level of the i-th consumer, and $x_i \in \mathcal{X}^*_i(p)$ is his tentative consumption.

2.2 Equilibrium market flows

In accordance to the previous notations, we eventually say that the *real market flow*

$$\widetilde{F} = \left((\widetilde{y}_k = \alpha_k y_k)_{k=1}^K, (\widetilde{x}_i = \beta_i x_i)_{i=1}^I \right)$$

is defined by the triple (p, F, γ). Here, $p \in \mathbb{R}^n_+$ is the vector of prices,

$$F = \left((y_k)_{k=1}^K, (x_i)_{i=1}^I \right) \in \prod_{k=1}^K \mathcal{Y}_k \times \prod_{i=1}^I \mathcal{X}_i$$

is the *tentative market flow*, and

$$\gamma = \left((\alpha_k)_{k=1}^K, (\beta_i)_{i=1}^I \right) \in [0, 1]^{K+I}$$

is the *proper system of participation levels* (w.r.t. p and F), i.e.

$$\alpha_k = \begin{cases} 1, & \text{if } \langle p, y_k \rangle - c_k(y_k) > \kappa_k, \\ 0, & \text{if } \langle p, y_k \rangle - c_k(y_k) < \kappa_k \end{cases}, \quad \beta_i = \begin{cases} 1, & \text{if } \langle p, x_i \rangle < w_i, \\ 0, & \text{if } \langle p, x_i \rangle > w_i. \end{cases}$$

Now we define the market equilibrium in the standard way.

Definition 1 (Equilibrium market flow) *We say that* $p^* \in \mathbb{R}^n$ *is the* equilibrium price *if there exists a real market flow*

$$\widetilde{F}^* = \left((\widetilde{y}_k^* = \alpha_k^* y_k^*)_{k=1}^K, (\widetilde{x}_i^* = \beta_i^* x_i^*)_{i=1}^I \right) \in \prod_{k=1}^K S_k(p^*) \times \prod_{i=1}^I D_i(p^*),$$

satisfying the market clearing condition

$$p^* \geq 0, \quad \sum_{k=1}^K \widetilde{y}_k^* - \sum_{i=1}^I \widetilde{x}_i^* \geq 0, \quad \left\langle p^*, \sum_{k=1}^K \widetilde{y}_k^* - \sum_{i=1}^I \widetilde{x}_i^* \right\rangle = 0. \tag{7}$$

In this case, $\widetilde{F}^*$ *is called the equilibrium market flow.*

The market clearing condition (7) states that the real consumption never exceed the real production, and the markets of goods with positive prices ($p^{(j)} > 0$) are perfectly cleared:

$$\sum_{k=1}^K \widetilde{y}_k^{*(j)} = \sum_{i=1}^I \widetilde{x}_i^{*(j)}.$$

3 Decentralization of prices

3.1 Characterization of equilibrium prices

Given a vector of prices $p \in \mathbb{R}^n_+$, producers maximize their profits and consumers minimize their expenditures. Afterwards, both properly adjust their participation levels by comparing the profits with the fixed costs, in case of producers, or by comparing the expenditures with the wealths, in case of consumers. Exactly the same behavior can be obtained by maximizing their excessive revenues.

The *excessive revenue* of the k-th producer is set as

$$ERP_k(p) \stackrel{\text{def}}{=} (\pi_k(p) - \kappa_k)_+ = \max_{y_k \in \mathcal{Y}_k} \left(\langle p, y_k \rangle - c_k(y_k) - \kappa_k \right)_+, \tag{8}$$

where

$$\mathcal{X}_i \stackrel{\text{def}}{=} \left\{ x_i \in X_i \,|\, u_i(x_i) \geq u_i \right\}.$$

Using the substitution $\widetilde{y}_k = \alpha_k y_k$, we obtain

$$ERP_k(p) = (\pi_k(p) - \kappa_k)_+ = \max_{\alpha_k \in [0,1]} \alpha_k \left(\pi_k(p) - \kappa_k \right) =$$

$$\max_{\substack{\alpha_k \in [0,1] \\ y_k \in \mathcal{Y}_k}} \alpha_k \left(\langle p, y_k \rangle - c_k(y_k) - \kappa_k\right) = \max_{\substack{\alpha_k \in [0,1] \\ \widetilde{y}_k \in \alpha_k \mathcal{Y}_k}} \langle p, \widetilde{y}_k \rangle - \alpha_k c_k \left(\widetilde{y}_k / \alpha_k\right) - \alpha_k \kappa_k.$$

Note that the maximization problem

$$ERP_k(p) = \max_{\substack{\alpha_k \in [0,1] \\ \widetilde{y}_k \in \alpha_k \mathcal{Y}_k}} \langle p, \widetilde{y}_k \rangle - \alpha_k c_k \left(\widetilde{y}_k / \alpha_k\right) - \alpha_k \kappa_k$$

is convex, and its set of optimal solutions consists of proper participation levels α_k and real production bundles $\widetilde{y}_k$. Moreover, $ERP_k(p)$ is convex in p as the maximum of linear functions. Hence, the convex subdifferential of the excessive revenue ERP_k gives the supply S_k of the k-th producer, i.e.

$$\partial ERP_k(p) = S_k(p). \tag{9}$$

The latter follows e.g. from [22, Theorem 2.4.18] on the convex subdifferential of a max-type function.

Analogously, we define the *excessive revenue* of the i-th consumer as follows:

$$ERC_i(p) \stackrel{\text{def}}{=} \left(w_i - e_i(p)\right)_+ = \max_{x_i \in \mathcal{X}_i} \left(w_i - \langle p, x_i \rangle\right)_+ . \tag{10}$$

Using the substitution $\widetilde{x}_i = \beta_i x_i$, we obtain

$$ERC_i(p) = \left(w_i - e_i(p)\right)_+ = \max_{\beta_i \in [0,1]} \beta_i \left(w_i - e_i(p)\right) =$$

$$\max_{\substack{\beta_i \in [0,1] \\ x_i \in \mathcal{X}_i}} \beta_i \left(w_i - \langle p, x_i \rangle\right) = \max_{\substack{\beta_i \in [0,1] \\ \widetilde{x}_i \in \beta_i \mathcal{X}_i}} \beta_i w_i - \langle p, \widetilde{x}_i \rangle .$$

Note that $\widetilde{x}_i \in \beta_i \mathcal{X}_i$ means

$$\widetilde{x}_i \in \beta_i X_i \text{ and } u_i \left(\widetilde{x}_i / \beta_i\right) \geq u_i.$$

In particular, $\beta_i u_i \left(\widetilde{x}_i / \beta_i\right)$ is jointly concave in $(\widetilde{x}_i, \beta_i)$. The maximization problem

$$ERC_i(p) = \max_{\substack{\beta_i \in [0,1] \\ \widetilde{x}_i \in \beta_i \mathcal{X}_i}} \beta_i w_i - \langle p, \widetilde{x}_i \rangle$$

is convex, and its set of optimal solutions consists of proper participation levels β_i and real consumption bundles $\widetilde{x}_i$. Moreover, $ERC_i(p)$ is convex in p as the maximum of linear functions. Hence, the convex subdifferential of the excessive revenue ERC_i gives the opposite demand D_i of the i-th consumer, i.e.

$$\partial ERC_i(p) = -D_i(p). \tag{11}$$

The latter follows also from [22, Theorem 2.4.18].

Overall, we define the *total excessive revenue* as the sum of excessive revenues of all agents:

$$TER(p) \stackrel{\text{def}}{=} \sum_{k=1}^{K} EPR_k(p) + \sum_{i=1}^{I} ECR_i(p). \tag{12}$$

Note that function $TER(\cdot)$ is convex since it is a sum of convex functions. Moreover, its convex subdifferential represents the excess supply due to (9) and (11).

By application of [19, Theorem 23.8] on the subdifferential of the sum of convex functions, we obtain:

Theorem 1 (Excess supply and TER, [16, 18]) *For $p \in \mathbb{R}^n_+$ it holds:*

$$\partial TER(p) = \sum_{k=1}^{K} S_k(p) - \sum_{i=1}^{I} D_i(p).$$

Theorem 1 allows us to characterize equilibrium prices as minimizers of TER.

Theorem 2 (Characterization of equilibrium prices, [16, 18]) *$p \in \mathbb{R}^n_+$ is a system of equilibrium prices if and only if it solves the following convex minimization problem:*

$$\min_{p \in \mathbb{R}^n_+} TER(p). \qquad (\mathbf{P})$$

3.2 Trade design

Theorem 2 reveals the origin of equilibrium prices at the market. Namely, in order to reach an equilibrium price one needs to solve the minimization problem ($\mathbf{P}$):

$$\min_{p \in \mathbb{R}^n_+} TER(p).$$

Our goal is to explain how agents can efficiently tackle this nonsmooth convex minimization problem by successively updating prices. It is crucial for our approach that the updates of prices correspond to subgradient-type schemes for solving ($\mathbf{P}$). Due to Theorem 1, the subgradients $\nabla TER(p)$ represent the *excess supply*, i.e.

$$\nabla TER(p) = \sum_{k=1}^{K} \widetilde{y}_k - \sum_{i=1}^{I} \widetilde{x}_i, \text{ where } \widetilde{y}_k \in S_k(p), \widetilde{x}_i \in D_i(p). \qquad (13)$$

This gives rise to use the subgradients $z(p)$ for the iterative minimization of TER. E.g., the change of prices Δp can be taken proportional to the current excess demand:

$$\Delta p \sim -\nabla TER(p).$$

However, as it can be seen from (13), the subgradients of TER are not known to individual agents. Indeed, $z(p)$ represents the *aggregate* excess supply. For getting access to its value, one would assume the existence of a manager who collects the information about agents' production and consumption bundles, and aggregates them over the whole market. Here, the full information about production and consumption over the market must be available to him. Besides, the prices need to be updated by the manager, thus, leading to price regulation. Clearly, these assumptions can be justified only within a centrally planned economy. Aiming to avoid this restriction, we decentralize prices.

The decentralization of prices can be implemented by the introduction of various price designs. In this paper, we focus just on the *trade design*:

k-th producer settles and updates his individual prices p_k,
and consumers buy at the lowest purchase price $\min\limits_{k=1,\ldots,K} p_k$.

Note that for vectors $p_1, \ldots, p_K \in \mathbb{R}^n$, we denote by $\min\limits_{k=1,\ldots,K} p_k \in \mathbb{R}^n$ the vector with coordinates

$$\left(\min_{k=1,\ldots,K} p_k \right)^{(j)} = \min_{k=1,\ldots,K} p_k^{(j)}, \quad j = 1, \ldots, K.$$

Now, the total excessive revenue depends on the producers' prices $(p_k)_{k=1}^K$ as follows:

$$TER(p_1, \ldots, p_K) \stackrel{\text{def}}{=} \sum_{k=1}^K EPR_k\left(p_k\right) + \sum_{i=1}^I ECR_i \left(\min_{k=1,\ldots,K} p_k \right) =$$

$$\sum_{k=1}^K \max_{y_k \in \mathcal{Y}_k} \left(\langle p, y_k \rangle - c_k(y_k) \right)_+ + \sum_{i=1}^I \max_{x_i \in \mathcal{X}_i} \left(w_i - \left\langle \min_{k=1,\ldots,K} p_k, x_i \right\rangle \right)_+ . \tag{14}$$

The decentralization of prices makes the corresponding subdifferential information about excess demands available to producers. In fact, note that the total excessive revenue TER from (14) is convex in the variables $(p_k)_{k=1}^K$. Let us obtain an expression for its convex subgradients $\nabla_{p_k} TER(p_1, \ldots, p_K)$ w.r.t. p_k:

$$\nabla_{p_k} TER(p_1, \ldots, p_K) = \widetilde{y}_k - \sum_{i=1}^I \mu_{ik} \circ \widetilde{x}_i, \quad k = 1, \ldots, K. \tag{15}$$

Here, $\widetilde{y}_k \in S_k(p_k)$ is the supply of k-th producer w.r.t. his individual price p_k, and $\widetilde{x}_i \in D_i \left(\min\limits_{k=1,\ldots,K} p_k \right)$ is the demand of i-th consumer w.r.t. the lowest purchase price $\min\limits_{k=1,\ldots,K} p_k$. Further, $\mu_{ik}^{(j)}$ denotes the share of i-th consumer's demand from k-th producer for good j. Indeed, the shares $\mu_{ik}^{(j)}$ for good j sum up to 1 over all producers $k = 1, \ldots, K$. Moreover, the share $\mu_{ik}^{(j)}$ vanishes if the k-th producer's price $p_k^{(j)}$ exceeds the lowest purchase price $\min\limits_{k=1,\ldots,K} p_k^{(j)}$ for good j. Thus, we write

$$\left(\mu_{ik}\right)_{k=1}^K \in M\left(p_1, \ldots, p_K\right),$$

where

$$M\left(p_1, \ldots, p_K\right) \stackrel{\text{def}}{=} \left\{ \left(\mu_k\right)_{k=1}^K \in [0,1]^{n \times K} \;\middle|\; \begin{array}{c} \sum\limits_{k=1}^K \mu_k^{(j)} = 1, \\ \mu_k^{(j)} = 0 \text{ if } p_k^{(j)} \neq \min\limits_{k=1,\ldots,K} p_k^{(j)} \\ j = 1, \ldots, n, k = 1, \ldots, K \end{array} \right\}.$$

We claim that the subdifferential information in (15) is known to k-th producer. First, note that $\widetilde{y}_k$ is his real production. Despite of the fact that the shares μ_{ik} and the demands $\widetilde{x}_i$ cannot be estimated by k-th producer, their aggregated product $\sum_{i=1}^I \mu_{ik} \circ \widetilde{x}_i$

is perfectly available to him. Indeed, $\sum_{i=1}^{I} \mu_{ik} \circ \widetilde{x}_i$ forms the bundle of goods demanded by all consumers from k-th producer. Altogether, the subgradients $\nabla_{p_k} TER(p_1, \ldots, p_K)$ represent the *individual excess* of k-th producer's supply over all consumers' demands. Overall, we obtain:

Theorem 3 (Producers' excess supply and TER)

$$\partial_{p_k} TER(p_1, \ldots, p_K) = S_k(p_k) - \sum_{i=1}^{I} \mu_{ik} \circ D_i \left(\min_{k=1,\ldots,K} p_k \right), \quad k = 1, \ldots, K,$$

with demand shares $(\mu_{ik})_{k=1}^{K} \in M(p_1, \ldots, p_K)$.

Due to Theorem 3, the subdifferential of $TER(p_1, \ldots, p_k)$ is completely available to k-th producer. This fact suggests to adjust prices by solving the minimization problem

$$\min_{p_1,\ldots,p_K \in \mathbb{R}^n_+} TER(p_1, \ldots, p_K). \tag{\textbf{PD}}$$

Note that the minimization problem (**PD**) is stated w.r.t. the decentralized producers' prices $(p_k)_{k=1}^{K}$, while previously in (**P**) one minimizes over the common prices p.

3.3 Adjoint problem

We relate the minimization problems (**P**) and (**PD**) by exploiting the fact that they have the same adjoint problem. In order to state the adjoint problem for (**P**) and (**PD**), we set

$$\alpha \stackrel{\text{def}}{=} \{\alpha_k\}_{k=1}^{K}, \widetilde{y} \stackrel{\text{def}}{=} \{\widetilde{y}_k\}_{k=1}^{K}, \beta \stackrel{\text{def}}{=} \{\beta_i\}_{i=1}^{I}, \widetilde{x} \stackrel{\text{def}}{=} \{\widetilde{x}_i\}_{i=1}^{I},$$
$$\mathcal{Y} \stackrel{\text{def}}{=} \prod_{k=1}^{K} \mathcal{Y}_k, \alpha\mathcal{Y} \stackrel{\text{def}}{=} \prod_{k=1}^{K} \alpha_k \mathcal{Y}_k, \mathcal{X} \stackrel{\text{def}}{=} \prod_{i=1}^{I} \mathcal{X}_i, \beta\mathcal{X} \stackrel{\text{def}}{=} \prod_{i=1}^{I} \beta_i \mathcal{X}_i.$$

Here, α, β represent participation levels, and $\widetilde{y}$, $\widetilde{x}$ represent real production and consumption bundles, respectively. Moreover, $\alpha\mathcal{Y}$, $\beta\mathcal{X}$ represent real production and real consumption sets given the participation levels α, β, respectively.

The feasible set of the adjoint problem is formed by participation levels and corresponding real production and consumption bundles, i.e.

$$\mathcal{A} \stackrel{\text{def}}{=} \left\{ (\alpha, \widetilde{y}, \beta, \widetilde{x}) \;\middle|\; \begin{array}{l} (\alpha, \widetilde{y}) \in [0,1]^K \times \alpha\mathcal{Y} \\ (\beta, \widetilde{x}) \in [0,1]^I \times \beta\mathcal{X} \end{array} \right\}.$$

Note that the set A is convex. Further, the following adjoint constraint need to be satisfied:

$$\sum_{k=1}^{K} \widetilde{y}_k \geq \sum_{i=1}^{I} \widetilde{x}_i, \tag{16}$$

meaning that the aggregate real consumption does not exceed the aggregate real production. The objective function of the adjoint problem is

$$\Phi(\alpha, \widetilde{y}, \beta, \widetilde{x}) \stackrel{\text{def}}{=} \sum_{i=1}^{I} \beta_i w_i - \sum_{k=1}^{K} \alpha_k c_k (\widetilde{y}_k / \alpha_k) + \alpha_k \kappa_k,$$

expressing the difference between the aggregate wealth spent for real consumption and producers' costs. Finally, we consider the maximization problem

$$\max_{(\alpha,\widetilde{y},\beta,\widetilde{x})\in\mathcal{A}} \left\{ \Phi\left(\alpha,\widetilde{y},\beta,\widetilde{x}\right) \;\middle|\; \sum_{k=1}^{K}\widetilde{y}_k \geq \sum_{i=1}^{I}\widetilde{x}_i \right\}. \tag{$\mathbf{A}$}$$

It turns out that ($\mathbf{A}$) is the adjoint problem for ($\mathbf{P}$) and ($\mathbf{PD}$). The proof of this fact uses the following simple Lemma 1.

Lemma 1 *For $y_k, x \in \mathbb{R}^n_+$, $k = 1,\ldots,K$, the inequality*

$$\sum_{k=1}^{K} y_k \geq x \tag{17}$$

is equivalent to

$$\sum_{k=1}^{K}\langle p_k, y_k\rangle \geq \left\langle \min_{k=1,\ldots,K} p_k, x\right\rangle \;\; \textit{for all } p_k \in \mathbb{R}^n_+, k=1,\ldots,K. \tag{18}$$

Proof:
(i) Let (17) be satisfied. For $p_k \in \mathbb{R}^n_+$, $k = 1,\ldots,K$, we have

$$\sum_{k=1}^{K}\langle p_k, y_k\rangle - \left\langle \min_{k=1,\ldots,K} p_k, x\right\rangle = \sum_{j=1}^{n}\left(\sum_{k=1}^{K} p_k^{(j)} y_k^{(j)} - \min_{k=1,\ldots,K} p_k^{(j)} x^{(j)}\right).$$

For (18) to hold, it is sufficient to show that

$$\sum_{k=1}^{K} p_k^{(j)} y_k^{(j)} - \min_{k=1,\ldots,K} p_k^{(j)} x^{(j)} \geq 0 \text{ for all } j = 1,\ldots,n.$$

Indeed, setting for fixed $j \in \{1,\ldots,n\}$

$$p^{(j)} = \min_{k=1,\ldots,K} p_k^{(j)} \text{ and } \mathcal{K}^{(j)} = \left\{k \in \{1,\ldots,K\} \;\middle|\; p_k^{(j)} = p^{(j)}\right\}, \tag{19}$$

we obtain:

$$\sum_{k=1}^{K} p_k^{(j)} y_k^{(j)} - \min_{k=1,\ldots,K} p_k^{(j)} x^{(j)} = \sum_{k\in\mathcal{K}^{(j)}} p^{(j)} y_k^{(j)} + \sum_{k\notin\mathcal{K}^{(j)}} p_k^{(j)} y_k^{(j)} - p^{(j)} x^{(j)} =$$

$$= \sum_{k\in\mathcal{K}^{(j)}} p^{(j)} y_k^{(j)} + \sum_{k\notin\mathcal{K}^{(j)}} p_k^{(j)} y_k^{(j)} - p^{(j)} x^{(j)} + \sum_{k\notin\mathcal{K}^{(j)}} p^{(j)} y_k^{(j)} - \sum_{k\notin\mathcal{K}^{(j)}} p^{(j)} y_k^{(j)}$$

$$= p^{(j)}\left(\sum_{k=1}^{K} y_k^{(j)} - x^{(j)}\right) + \sum_{k\notin\mathcal{K}^{(j)}}\left(p_k^{(j)} - p^{(j)}\right) y_k^{(j)}.$$

The last expression is nonnegative due to (17), (19), and $p^{(j)}, y_k^{(j)} \in \mathbb{R}_+$, $k = 1, \ldots, K$.
(ii) Let (18) be satisfied. Setting there $p_k = p \in \mathbb{R}^n_+$, we get

$$\left\langle p, \sum_{k=1}^{K} x_k \right\rangle \geq \langle p, y \rangle \quad \text{for all } p \in \mathbb{R}^n_+.$$

Hence, (17) is fulfilled. □

Theorem 4 *It holds:*

$$\min_{p \in \mathbb{R}^n_+} TER(p) = \min_{p_1,\ldots,p_K \in \mathbb{R}^n_+} TER(p_1, \ldots, p_K)$$

$$= \max_{(\alpha, \widetilde{y}, \beta, \widetilde{x}) \in \mathcal{A}} \left\{ \Phi\left(\alpha, \widetilde{y}, \beta, \widetilde{x}\right) \;\middle|\; \sum_{k=1}^{K} \widetilde{y}_k \geq \sum_{i=1}^{I} \widetilde{x}_i \right\}.$$

Proof:

$$TER(p_1, \ldots, p_K) = \sum_{k=1}^{K} \max_{y_k \in \mathcal{Y}_k} \left(\langle p, y_k \rangle - c_k(y_k) - \kappa_k\right)_+ + \sum_{i=1}^{I} \max_{x_i \in \mathcal{X}_i} \left(w_i - \left\langle \min_{k=1,\ldots,K} p_k, x_i \right\rangle \right)_+$$

$$= \sum_{k=1}^{K} \max_{\substack{\alpha_k \in [0,1] \\ y_k \in \mathcal{Y}_k}} \alpha_k \left(\langle p, y_k \rangle - c_k(y_k) - \kappa_k\right) + \sum_{i=1}^{I} \max_{\substack{\beta_i \in [0,1] \\ x_i \in \mathcal{X}_i}} \beta_i \left(w_i - \left\langle \min_{k=1,\ldots,K} p_k, x_i \right\rangle \right)$$

$$= \max_{\substack{(\alpha, y) \in [0,1]^K \times \mathcal{Y} \\ (\beta, x) \in [0,1]^I \times \mathcal{X}}} \sum_{k=1}^{K} \alpha_k \left(\langle p, y_k \rangle - c_k(y_k) - \kappa_k\right) + \sum_{i=1}^{I} \beta_i \left(w_i - \left\langle \min_{k=1,\ldots,K} p_k, x_i \right\rangle \right)$$

$$= \max_{\substack{(\alpha, \widetilde{y}) \in [0,1]^K \times \alpha\mathcal{Y} \\ (\beta, \widetilde{x}) \in [0,1]^I \times \beta\mathcal{X}}} \sum_{k=1}^{K} \left(\langle p, \widetilde{y}_k \rangle - \alpha_k c_k\left(\widetilde{y}_k / \alpha_k\right) - \alpha_k \kappa_k\right) + \sum_{i=1}^{I} \left(\beta_i w_i - \left\langle \min_{k=1,\ldots,K} p_k, \widetilde{x}_i \right\rangle \right)$$

$$= \max_{(\alpha, \widetilde{y}, \beta, \widetilde{x}) \in \mathcal{A}} \Phi\left(\alpha, \widetilde{y}, \beta, \widetilde{x}\right) + \sum_{k=1}^{K} \langle p_k, \widetilde{y}_k \rangle - \left\langle \min_{k=1,\ldots,K} p_k, \sum_{i=1}^{I} \widetilde{x}_i \right\rangle. \tag{20}$$

Using this representation of $TER(p_1, \ldots, p_K)$, we obtain:

$$\min_{p_1,\ldots,p_K \in \mathbb{R}^n_+} TER(p_1, \ldots, p_K) =$$

$$= \min_{p_1,\ldots,p_K \in \mathbb{R}^n_+} \max_{(\alpha, \widetilde{y}, \beta, \widetilde{x}) \in \mathcal{A}} \Phi\left(\alpha, \widetilde{y}, \beta, \widetilde{x}\right) + \sum_{k=1}^{K} \langle p_k, \widetilde{y}_k \rangle - \left\langle \min_{k=1,\ldots,K} p_k, \sum_{i=1}^{I} \widetilde{x}_i \right\rangle$$

$$= \max_{(\alpha, \widetilde{y}, \beta, \widetilde{x}) \in \mathcal{A}} \Phi\left(\alpha, \widetilde{y}, \beta, \widetilde{x}\right) + \min_{p_1,\ldots,p_K \in \mathbb{R}^n_+} \sum_{k=1}^{K} \langle p_k, \widetilde{y}_k \rangle - \left\langle \min_{k=1,\ldots,K} p_k, \sum_{i=1}^{I} \widetilde{x}_i \right\rangle \tag{21}$$

$$= \max_{(\alpha,\widetilde{y},\beta,\widetilde{x})\in\mathcal{A}} \left\{ \Phi\left(\alpha,\widetilde{y},\beta,\widetilde{x}\right) \;\middle|\; \begin{array}{l} \sum\limits_{k=1}^{K} \langle p_k, \widetilde{y}_k\rangle \geq \left\langle \min\limits_{k=1,\ldots,K} p_k, \sum\limits_{i=1}^{I} \widetilde{x}_i \right\rangle \\ \text{for all } p_k \in \mathbb{R}^n_+, k=1,\ldots,K \end{array} \right\}.$$

Applying Lemma 1, the adjoint constraint (16) is equivalent to

$$\sum_{k=1}^{K} \langle p_k, \widetilde{y}_k\rangle \geq \left\langle \min_{k=1,\ldots,K} p_k, \sum_{i=1}^{I} \widetilde{x}_i \right\rangle \text{ for all } p_k \in \mathbb{R}^n_+, k=1,\ldots,K.$$

Overall, (**A**) is the adjoint problem for (**PD**).

Analogously, (**A**) is the adjoint problem for (**P**). Instead of Lemma 1, one uses here the equivalence of (16) to

$$\left\langle p, \sum_{k=1}^{K} \widetilde{y}_k \right\rangle \geq \left\langle p, \sum_{i=1}^{I} \widetilde{x}_i \right\rangle \text{ for all } p \in \mathbb{R}^n_+.$$

□

Corollary 1 *Let* $(p_k)_{k=1}^{K}$ *solve* (**PD**) *and* $(\alpha,\widetilde{y},\beta,\widetilde{x})$ *solve its adjoint problem* (**A**)*. Then, the lowest purchase prices* $\min\limits_{k=1,\ldots,K} p_k$ *form equilibrium prices with the proper system of participation levels* $\gamma = (\alpha,\beta)$ *and the corresponding equilibrium market flow* $\widetilde{F} = (\widetilde{y},\widetilde{x})$*. Moreover, the k-th producer's real production bundle* $\widetilde{y}_k^{(j)}$ *vanishes if his individual price* $p_k^{(j)}$ *exceeds the lowest purchase price* $\min\limits_{k=1,\ldots,K} p_k^{(j)}$ *for good j, i.e.*

$$\widetilde{y}_k^{(j)} = 0 \text{ if } p_k^{(j)} \neq \min_{k=1,\ldots,K} p_k^{(j)}, \quad k=1,\ldots,K, j=1,\ldots,n.$$

Proof:

Due to Theorem 4:

$$0 \leq TER\left(\min_{k=1,\ldots,K} p_k\right) - \Phi\left(\alpha,\widetilde{y},\beta,\widetilde{x}\right) \overset{(14)}{\leq} TER\left(p_1,\ldots,p_K\right) - \Phi\left(\alpha,\widetilde{y},\beta,\widetilde{x}\right) = 0.$$

Hence, $\min\limits_{k=1,\ldots,K} p_k$ solves (**P**). By Theorem 2, $\min\limits_{k=1,\ldots,K} p_k$ forms the system of equilibrium prices. Moreover, its proper system of participation levels is (α,β) and the corresponding equilibrium market flow is $(\widetilde{y},\widetilde{x})$. The latter follows from the fact that (**A**) is the adjoint problem for (**P**).

Further, (21) from Theorem 4 yields

$$\sum_{k=1}^{K} \langle p_k, \widetilde{y}_k\rangle - \left\langle \min_{k=1,\ldots,K} p_k, \sum_{i=1}^{I} \widetilde{x}_i \right\rangle = \sum_{k=1}^{K} \left\langle \min_{k=1,\ldots,K} p_k, \widetilde{y}_k \right\rangle - \left\langle \min_{k=1,\ldots,K} p_k, \sum_{i=1}^{I} \widetilde{x}_i \right\rangle = 0.$$

Thus,

$$\sum_{k=1}^{K} \left\langle p_k - \min_{k=1,\ldots,K} p_k, \widetilde{y}_k \right\rangle = 0,$$

or, equivalently,

$$\left\langle p_k^{(j)} - \min_{k=1,\dots,K} p_k^{(j)}, \widetilde{y}_k^{(j)} \right\rangle = 0, \quad k = 1, \dots, K, \quad j = 1, \dots, n.$$

The latter implies: $\widetilde{y}_k^{(j)} = 0$ if $p_k^j \neq \min\limits_{k=1,\dots,K} p_k^{(j)}$. $\square$

4 Price adjustment

We describe how producers may efficiently adjust their individual prices $(p_k)_{k=1}^K$ to arrive at a system of equilibrium prices. This price adjustment corresponds to the quasi-monotone subgradient method (SM) [17], which is described in Appendix for reader's convenience. It is applied to the minimization of the total excessive revenue (**PD**):

$$\min_{p_1,\dots,p_K \in \mathbb{R}^n_+} TER(p_1, \dots, p_K).$$

Let k-th producer choose a sequence of positive sensitivity parameters $\{\chi_k[t]\}_{t\geq 0}$, $k = 1, \dots, K$. We consider the following iteration:

Price Adjustment (PA)

1. Producers determine their current excess supplies $\nabla_{p_k} TER(p_1[t], \dots, p_K[t])$:

a) k-th producer computes optimal tentative production bundles

$$y_k(p_k[t]) \in \mathcal{Y}_k^*(p_k[t]),$$

and participation level

$$\alpha_k(p_k[t]) = \begin{cases} 1, & \text{if } \pi_k(p_k[t]) \geq \kappa_k, \\ 0, & \text{if } \pi_k(p_k[t]) < \kappa_k, \end{cases}$$

indicating whether $y_k(p_k[t])$ is worth to be implemented.
His real production bundle is $\alpha_k(p_k[t]) y_k(p_k[t])$, i.e. either $y_k(p_k[t])$ or zero.

b) i-th consumer identifies the lowest purchase prices

$$p[t] = \min_{k=1,\dots,K} p_k[t].$$

He computes optimal tentative consumption bundle

$$x_i(p[t]) \in \mathcal{X}_i^*(p[t]),$$

and participation level

$$\beta_i(p[t]) = \begin{cases} 1, & \text{if } e_i(p[t]) \leq w_i, \\ 0, & \text{if } e_i(p[t]) < w_i, \end{cases}$$

indicating whether $x_i(p[t])$ is implemented.
His real consumption bundle is $\beta_i(p[t]) x_i(p[t])$, i.e. either $x_i(p[t])$ or zero.

c) i-th consumer decides on demand shares

$$(\mu_{ik}[t])_{k=1}^{K} \in M\left(p_1[t], \dots, p_K[t]\right),$$

and demands from k-th producer the consumption bundle

$$\mu_{ik}[t] \circ \beta_i(p[t]) x_i(p[t]), \quad k = 1, \dots, K.$$

d) k-th producer computes his current excess supply

$$\nabla_{p_k} TER(p_1[t], \dots, p_K[t]) = \alpha_k(p_k[t]) y_k(p_k[t]) - \sum_{i=1}^{I} \mu_{ik}[t] \circ \beta_i(p[t]) x_i(p[t]). \quad (22)$$

2. k-th producer accumulates his excess supplies

$$z_k[t] = z_k[t-1] + \nabla_{p_k} TER(p_1[t], \dots, p_K[t]), \quad z_k[-1] = 0. \quad (23)$$

3. k-th producer computes the price forecast w.r.t. his sensitivity parameter $\chi_k[t]$

$$p_k^+[t] = \frac{1}{\chi_k[t]} \left(-z_k[t]\right)_+. \quad (24)$$

4. k-th producer updates

$$p_k[t+1] = \frac{t+1}{t+2} p_k[t] + \frac{1}{t+2} p_k^+[t] \quad (25)$$

by combining his previous price with his forecast. □

4.1 Price forecast and price update

First, we give an interpretation for the price forecast (24). Recall that $z_k[t]$ represents the excess of k-th producer's supply over consumers' demands for good j accumulated up to time t. If $z_k^{(j)}[t] \geq 0$, i.e. supply exceeds demand, then $p_k^{+(j)}[t] = 0$ for good j. In case of $z_k^{(j)}[t] < 0$, the price forecast $p_k^{+(j)}[t]$ is proportional to the accumulated individual excess demand of k-th producer. Here, $\chi_k[t]$ plays the role of a sensitivity parameter.

The price forecast (24) can be obtained within the Euclidean setup as follows. Let us denote the Euclidean prox-function for $\mathbb{R}^n_+$ and the corresponding Bregman distance as as

$$d(p) = \frac{1}{2} \sum_{j=1}^{n} \left(p^{(j)}\right)^2, \quad B(p, \bar{p}) = \frac{1}{2} \sum_{j=1}^{n} \left(p^{(j)} - \bar{p}^{(j)}\right)^2.$$

For $z \in \mathbb{R}^n, \chi > 0$ we consider the auxiliary minimization problem

$$\min_{p \in \mathbb{R}^n_+} \left\{ \langle z, p \rangle + \chi d(p) \right\} = -\frac{1}{2\chi} \sum_{j=1}^{n} \left(-z^{(j)}\right)_+^2. \quad (26)$$

Its unique solution is $p^+ = \frac{1}{\chi}(-z)_+$. The price forecast (24) reads then:

$$p_k^+[t] = \arg\min_{p \in \mathbb{R}^n_+} \left\{ \langle z_k[t], p \rangle + \chi_k[t] d(p) \right\}. \tag{27}$$

Secondly, let us interpret the price update (25):

$$p_k[t+1] = \frac{t+1}{t+2} p_k[t] + \frac{1}{t+2} p_k^+[t].$$

Due to the latter, the next price is a convex combination of the previous price and the price forecast. With time advancing, the proportion of the previous price becomes nearly one, but the fraction of the forecast vanishes. Hence, we conclude that our price update corresponds to a behavior of an experienced producer. He credits his experience much more than the current forecast. Further, from (25) we have

$$p_k[t+1] = \frac{1}{t+2} \left(p_k[0] + \sum_{r=0}^{t} p_k^+[r] \right). \tag{28}$$

The latter means that the prices generated by (PA) can be viewed as historical averages of preceding forecasts. This averaging pattern is also quite natural to assume for producer's behavior while adjusting prices.

4.2 Adjoint averaging

Our goal is to produce a feasible sequence for the adjoint problem ($\mathbf{A}$) by averaging participation levels, real production and consumption bundles from (PA). Along with the prices $\{(p_1[t], \ldots, p_K[t])\}_{t \geq 0}$ generated by method (PA), we consider the corresponding historical averages of participation levels:

$$\alpha_k[t] \stackrel{\text{def}}{=} \frac{1}{t+1} \sum_{r=0}^{t} \alpha_k(p_k[r]), \quad \beta_i[t] \stackrel{\text{def}}{=} \frac{1}{t+1} \sum_{r=0}^{t} \beta_i(p[r]).$$

Note that $\alpha_k[t] \in [0,1]$ is the frequency of successful production attempts by k-th producer up to time t. Analogously, $\beta_i[t] \in [0,1]$ is the frequency of successful consumption attempts by i-th consumer up to time t. We denote by

$$\gamma[t] = (\alpha[t], \beta[t]) \stackrel{\text{def}}{=} \left(\{\alpha_k[t]\}_{k=1}^K, \{\beta_i[t]\}_{i=1}^I \right)$$

the system of average participation levels. The historical averages of real production and consumption bundles are defined as follows:

$$\widetilde{y}_k[t] \stackrel{\text{def}}{=} \frac{1}{t+1} \sum_{r=0}^{t} \alpha_k(p_k[r]) y_k(p_k[r]), \quad \widetilde{x}_i[t] \stackrel{\text{def}}{=} \frac{1}{t+1} \sum_{r=0}^{t} \beta_i(p[r]) x_i(p[r]).$$

Due to convexity, $\widetilde{y}_k[t] \in \alpha_k[t] \mathcal{Y}_k$ and $\widetilde{x}_i[t] \in \beta_i[t] \mathcal{X}_i$. We denote by

$$\widetilde{F}[t] = (\widetilde{y}[t], \widetilde{x}[t]) \stackrel{\text{def}}{=} \left(\{\widetilde{y}_k[t]\}_{k=1}^K, \{\widetilde{x}_i[t]\}_{i=1}^I \right)$$

the average real market flow. Overall, the sequence

$$(\alpha[t], \widetilde{y}[t], \beta[t], \widetilde{x}[t]) \in \mathcal{A}, \quad t \geq 0,$$

is feasible for the adjoint problem (**A**).

Next Lemma 2 estimates the dual gap for the minimization problem (**PD**) and its adjoint problem (**A**) evaluated at the historical averages. For that, we set

$$\begin{array}{lcl} H_t(z) & \stackrel{\text{def}}{=} & -\min\limits_{p \in \mathbb{R}^n_+} \left\{ \langle p, z \rangle + \dfrac{1}{t+1} \sum\limits_{k=1}^{K} \chi_k[t] d(p) \right\}, \quad z \in \mathbb{R}^n, \\ R_t & \stackrel{\text{def}}{=} & -\sum\limits_{k=0}^{K} \sum\limits_{r=0}^{t} \min\limits_{p \in \mathbb{R}^n_+} \left\{ \begin{array}{l} \left\langle \nabla_{p_k} TER(p_1[r], \ldots, p_K[r]), p - p_k^+[r-1] \right\rangle \\ + \chi_k[r-1] B_d\left(p, p_k^+[r-1]\right) \end{array} \right\}. \end{array}$$

Lemma 2 *Let the sequence $\{p_1[t], \ldots, p_K[t]\}_{t \geq 0}$ be generated by (PA) with nondecreasing parameters*

$$\chi_k[t+1] \geq \chi_k[t], \quad t \geq 0, k = 1, \ldots, K.$$

Then, for all $t \geq 0$ it holds:

$$TER(p_1[t], \ldots, p_K[t]) - \Phi\left(\alpha[t], \widetilde{y}[t], \beta[t], \widetilde{x}[t]\right) + H_t\left(\sum_{k=1}^{K} \widetilde{y}_k[t] - \sum_{i=1}^{I} \widetilde{x}_i[t]\right) \leq \frac{R_t}{t+1}. \quad (29)$$

Proof:

It follows from (20) that the total excessive revenue is representable as a maximum of concave functions:

$$TER(p_1, \ldots, p_K) = \max_{(\alpha, \widetilde{y}, \beta, \widetilde{x}) \in \mathcal{A}} \Phi\left(\alpha, \widetilde{y}, \beta, \widetilde{x}\right) + \varphi\left(p_1, \ldots, p_K, \widetilde{y}, \widetilde{x}\right),$$

where

$$\varphi\left(p_1, \ldots, p_K, \widetilde{y}, \widetilde{x}\right) \stackrel{\text{def}}{=} \sum_{k=1}^{K} \langle p_k, \widetilde{y}_k \rangle - \left\langle \min_{k=1,\ldots,K} p_k, \sum_{i=1}^{I} \widetilde{x}_i \right\rangle.$$

Hence, we apply Theorem 6 and Lemma 5 from Appendix to get the following inequality:

$$TER(p_1[t], \ldots, p_K[t]) - \Phi\left(\alpha[t], \widetilde{y}[t], \beta[t], \widetilde{x}[t]\right)$$

$$-\min_{p_1, \ldots, p_K \in \mathbb{R}^n_+} \left\{ \varphi\left(p_1, \ldots, p_K, \widetilde{y}[t], \widetilde{x}[t]\right) + \frac{1}{t+1} \sum_{k=1}^{K} \chi_k[t] d(p_k) \right\} \leq \frac{R_t}{t+1}, \quad t \geq 0. \quad (30)$$

We estimate

$$\begin{aligned} & \min_{p_1, \ldots, p_K \in \mathbb{R}^n_+} \left\{ \varphi\left(p_1, \ldots, p_K, \widetilde{y}[t], \widetilde{x}[t]\right) + \frac{1}{t+1} \sum_{k=1}^{K} \chi_k[t] d(p_k) \right\} \\ & \leq \min_{p \in \mathbb{R}^n_+} \left\{ \varphi\left(p, \ldots, p, \widetilde{y}[t], \widetilde{x}[t]\right) + \frac{1}{t+1} \sum_{k=1}^{K} \chi_k[t] d(p) \right\} \\ & = \min_{p \in \mathbb{R}^n_+} \left\{ \left\langle p, \sum_{k=1}^{K} \widetilde{y}_k[t] - \sum_{i=1}^{I} \widetilde{x}_i[t] \right\rangle + \frac{1}{t+1} \sum_{k=1}^{K} \chi_k[t] d(p) \right\} \end{aligned}$$

Altogether, (30) implies (29). □

4.3 Convergence properties of (PA)

In order to arrive at the equilibrium price, producers need to appropriately adjust their sensitivity parameters $\{\chi_k[t]\}_{t\geq 0}$, $k = 1, \ldots, K$. In this Section we identify successful adjustment strategies of sensitivity parameters. Aiming at estimating the remainder term R_t from Lemma 2, we note that the sequence of k-th producer's excess supplies $\{\nabla_{p_k} TER(p_1[t], \ldots, p_K[t])\}_{t\geq 0}$ is bounded w.r.t. Euclidean norm, i.e.

$$\|\nabla_{p_k} TER(p_1[t], \ldots, p_K[t])\| \leq L, \quad t \geq 0, k = 1, \ldots, K. \tag{31}$$

The existence of the constant $L > 0$ in (31) follows from the compactness of production sets $\mathcal{Y}_k$, $k = 1, \ldots, K$, and consumption sets $\mathcal{X}_i$, $i = 1, \ldots, I$ (see Section 2).

We estimate the penalty term H_t and the remainder term R_t from Lemma 2.

Lemma 3 *It holds for $t \geq 0$:*

(i) $H_t(z) = \dfrac{t+1}{2\sum_{k=1}^K \chi_k[t]} \displaystyle\sum_{j=1}^n \left(-z^{(j)}\right)_+^2, \quad z \in \mathbb{R}^n,$

(ii) $R_t \leq \dfrac{L^2}{2} \displaystyle\sum_{k=1}^K \sum_{r=0}^t \frac{1}{\chi_k[r-1]}.$

Proof:
(i) follows immediately from (26).
(ii) For $g \in \mathbb{R}^n, p^+ \in \mathbb{R}^n_+, \chi > 0$ we estimate

$$\min_{p\in\mathbb{R}^n_+} \left\{\left\langle g, p - p^+\right\rangle + \chi B_d\left(p, p^+\right)\right\} \geq$$

$$\min_{p\in\mathbb{R}^n} \left\{\left\langle g, p - p^+\right\rangle + \chi B_{d_k}\left(p, p^+\right)\right\} = -\frac{1}{2\chi} \sum_{j=1}^n \left(g^{(j)}\right)^2.$$

Using the latter inequality, we get:

$$R_t \leq \sum_{k=1}^K \sum_{r=0}^t \frac{1}{2\chi_k[r-1]} \sum_{j=1}^n \left(\nabla^{(j)}_{p_k} TER(p_1[r], \ldots, p_K[r])\right)^2.$$

Due to (31), it follows:

$$R_t \leq \sum_{k=1}^K \sum_{r=0}^t \frac{L^2}{2\chi_k[r-1]}.$$

□

Next Lemma 4 clarifies the appropriate choice of producers' sensitivity parameters $\{\chi_k[t]\}_{t\geq 0}$, $k = 1, \ldots, K$. The indicator function of $\mathbb{R}^n_+$ is denoted by

$$\delta(z) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } z \in \mathbb{R}^n_+, \\ \infty, & \text{else.} \end{cases}$$

Lemma 4 *Let producers apply in (PA) sensitivity parameters satisfying*

$$\frac{\chi_k[t]}{t+1} \to 0, \quad \frac{\chi_k[t]}{\sqrt{t}} \to \infty, \quad k = 1, \dots, K.$$

Then,

$$H_t(\cdot) \to \delta(\cdot), \text{ and } \frac{R_t}{t+1} \to 0.$$

Moreover, the achievable order of convergence is $O\left(\frac{1}{\sqrt{t}}\right)$.

Proof:
If $z \in \mathbb{R}^n_+$, then $H_t(z) = 0$. Otherwise, $H^1_t(z) \to \infty$ as $\frac{t+1}{\sum_k^K \chi_k[t]} \to \infty$. Here, the order of convergence is $\frac{\sum_{k=1}^K \chi_k[t]}{t+1}$.

Due to $\frac{\chi_k[t]}{\sqrt{t}} \to \infty$, there exists $T > 0$ such that

$$\chi_k[t] \geq \sqrt{t}, \quad t \geq T, k = 1, \dots, K.$$

Using Lemma 3 we obtain:

$$\frac{R_t}{t+1} \leq \frac{L^2}{2} \sum_{k=1}^K \left[\frac{1}{t+1} \sum_{r=0}^T \frac{1}{\chi_k[r-1]} + \frac{1}{t+1} \sum_{r=T}^t \frac{1}{\sqrt{r}} \right].$$

Immediately, we see that $\frac{1}{t+1} \sum\limits_{r=0}^T \frac{1}{\chi_k[r-1]} \to 0$. Note that for a convex univariate function $\xi(r)$, $r \in \mathbb{R}$, and integer bounds a, b, we have

$$\sum_{r=a}^b \xi(r) \leq \int\limits_{a-1/2}^{b+1/2} \xi(s) \mathrm{d}s. \tag{32}$$

Hence, we also have

$$\frac{1}{t+1} \sum_{r=T}^t \frac{1}{\sqrt{r}} \overset{(32)}{\leq} \frac{1}{t+1} \int\limits_{T-1/2}^{t+1/2} \frac{1}{\sqrt{s}} \mathrm{d}s$$

$$= \frac{2}{t+1} \sqrt{s} \Big|_{T-1/2}^{t+1/2} = \frac{2}{t+1} \left(\sqrt{t+1/2} - \sqrt{T-1/2} \right) \to 0.$$

Here, the order of convergence is $O\left(\frac{1}{\sqrt{t}}\right)$.

Overall, the order of convergence is

$$\max \left\{ \frac{\sum_{k=1}^K \chi_k[t]}{t+1}, O\left(\frac{1}{\sqrt{t}}\right) \right\}. \tag{33}$$

Choosing $\chi_k[t] = O(\sqrt{t})$, $k = 1, \dots, K$, we achieve $O\left(\frac{1}{\sqrt{t}}\right)$ in (33). $\square$

Now, we are ready to prove the main convergence result for (PA) in Euclidean setup.

Theorem 5 *Let producers apply in (PA) sensitivity parameters satisfying*

$$\frac{\chi_k[t]}{t+1} \to 0, \quad \frac{\chi_k[t]}{\sqrt{t}} \to \infty, \quad k = 1, \ldots, K.$$

Then, the sequence $(p_k[t])_{k=1}^K$ converge to an equilibrium price. Moreover, the corresponding proper system of participation levels is approached by the system of average participation levels $\gamma[t] = (\alpha[t], \beta[t])$. The corresponding equilibrium market flow is approached by the average real market flow $\widetilde{F}[t] = (\widetilde{y}[t], \widetilde{x}[t])$. The achievable rate of convergence is of the order $O\left(\frac{1}{\sqrt{t}}\right)$.

Proof:
From Lemma 2 we obtain:

$$TER(p_1[t], \ldots, p_K[t]) - \Phi\left(\alpha[t], \widetilde{y}[t], \beta[t], \widetilde{x}[t]\right) + H_t\left(\sum_{k=1}^K \widetilde{y}_k[t] - \sum_{i=1}^I \widetilde{x}_i[t]\right) \le \frac{R_t}{t+1}. \quad (34)$$

The left-hand side of this inequality is composed by the objective function TER of the primal problem (**PD**), computed at the current prices $(p_1[t], \ldots, p_K[t])$, objective function Φ of its adjoint problem (**A**), computed at historical averages $(\alpha[t], \widetilde{y}[t], \beta[t], \widetilde{x}[t])$, and the quadratic penalty H_t for violation of the adjiont constraint:

$$\sum_{k=1}^K \widetilde{y}_k[t] \ge \sum_{i=1}^I \widetilde{x}_i[t], \quad k = 1, \ldots, K.$$

Due to the choice of sensitivity parameters $\chi_k[t]$, $k = 1, \ldots, K$, Lemma 4 provides:

$$H_t(\cdot) \to \delta(\cdot), \text{ and } \frac{R_t}{t+1} \to 0.$$

Hence, $(p_k[t])_{k=1}^K$ converges towards a solution of (**PD**), and $(\alpha[t], \widetilde{y}[t], \beta[t], \widetilde{x}[t])$ converges towards a solution of (**A**) by order $O\left(\frac{1}{\sqrt{t}}\right)$. Finally, we apply Corollary 1 to obtain the assertion. □

Now, we turn our attention to the case of constant sensitivity parameters.

Remark 1 *Let producers apply in (PA) constant sensitivity parameters. Let $\varepsilon > 0$ denote the tolerance for convergence of $(p_k[t])_{k=1}^K$ towards a solution of the primal problem (**PD**), and $(\alpha[t], \widetilde{y}[t], \beta[t], \widetilde{x}[t])$ towards a solution of its adjoint problem (**A**). Our goal is to indicate the number of steps $t(\varepsilon)$ and the sensitivity parameters $\chi_k(\varepsilon)$, $k = 1 \ldots, K$, in order to guarantee the tolerance ε for this primal-adjoint process. For that, we apply constant sensitivity parameters $\chi_k[t] = \chi_k$, $k = 1 \ldots, K$ to obtain*

$$H_t(z) = \frac{t+1}{2\sum_{k=1}^K \chi_k} \sum_{j=1}^n \left(-z^{(j)}\right)_+^2, \quad \frac{R_t}{t+1} \le \frac{L^2}{2} \sum_{k=1}^K \frac{1}{\chi_k}.$$

Recalling (34), the order of convergence for the primal-adjoint process is

$$\max\left\{\frac{\sum_{k=1}^{K}\chi_k}{t+1}, \sum_{k=1}^{K}\frac{1}{\chi_k}\right\}.$$

Choosing

$$t(\varepsilon) = O\left(\frac{1}{\varepsilon^2}\right), \chi_k(\varepsilon) = O\left(\frac{1}{\varepsilon}\right), \quad k = 1, \ldots, K,$$

we have

$$\max\left\{\frac{\sum_{k=1}^{K}\chi_k(\varepsilon)}{t(\varepsilon)+1}, \sum_{k=1}^{K}\frac{1}{\chi_k(\varepsilon)}\right\} = O(\varepsilon).$$

□

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Appendix

Quasi-monotone subgradient methods

Following the approach from [17], we present a quasi-monotone subgradient method for nonsmooth convex minimization. We consider the following minimization problem:

$$\min_{x \in X} f(x), \tag{35}$$

where $X \subset \mathbb{R}^n$ is a closed convex set with nonempty interior $\operatorname{int} X$, and f is a convex function on $\mathbb{R}^n$.

For function f, we denote by $\nabla f(x)$ its arbitrary subgradient at $x \in X$:

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle, \quad y \in X. \tag{36}$$

For the set X, we assume to be known a prox-function $d(x)$.

Definition 2 *$d : X \mapsto \mathbb{R}$ is called a prox-function for X w.r.t. $Y \subset X$ if the following holds:*

- *$d(x) \geq 0$ for all $x \in X$ and $d(x[0]) = 0$ for certain $x[0] \in Y$;*
- *d is continuously differentiable on Y and convex on X;*
- *Auxiliary minimization problem*

$$\min_{x \in X} \{\langle z, x\rangle + \chi d(x)\} \tag{37}$$

is easily solvable for $z \in \mathbb{R}^n, \chi > 0$, and admits the unique minimizer $x(z, \chi) \in Y$.

Given a prox-function d, we define the corresponding Bregman distance

$$B_d(x, y) \overset{\text{def}}{=} d(x) - d(y) - \langle x - y, \nabla d(y)\rangle \quad \text{for } x \in X, y \in Y. \tag{38}$$

Due to convexity of d on X, $B_d(x, y) \geq 0$. Moreover, for $x \in X$ it holds:

$$d(x) = d(x[0]) + \langle x - x[0], \nabla d(x[0])\rangle + B_d(x, x[0]) \geq B_d(x, x[0]). \tag{39}$$

For a sequence of positive parameters $\{\chi[t]\}_{t \geq 0}$, we consider the following iteration:

Quasi-monotone Subgradient Method (SM)

1. Take a current subgradient $\nabla f(x[t])$.

2. Accumulate subgradients $z[t] = z[t-1] + \nabla f(x[t])$, $z[-1] = 0$.

3. Compute the forecast $x^+[t] = \arg\min\limits_{x \in X} \{\langle z[t], x\rangle + \chi[t] d(x)\}$.

4. Update by combining $x[t+1] = \dfrac{t+1}{t+2} x[t] + \dfrac{1}{t+2} x^+[t]$.

Note that from (SM) we have

$$z[t] = \sum_{r=0}^{t} \nabla f(x[r]), \quad x[t+1] = \frac{1}{t+2}\left(x[0] + \sum_{r=0}^{t} x^+[r]\right).$$

Further, we set for $t \geq 0$:

$$\begin{aligned}
\ell_t(x) &\overset{\text{def}}{=} \sum_{r=0}^{t} f(x[r]) + \langle \nabla f(x[r]), x - x[r]\rangle, \\
\psi_t^* &\overset{\text{def}}{=} \min_{x \in X} \{\ell_t(x) + \chi[t] d(x)\}, \\
R_t &\overset{\text{def}}{=} -\sum_{r=0}^{t} \min_{x \in X} \left\{\left\langle \nabla f(x[r]), x - x^+[r-1]\right\rangle + \chi[r-1] B_d\left(x, x^+[r-1]\right)\right\},
\end{aligned}$$

where $x^+[-1] = x[0], \chi[-1] = \chi[0]$.

Next result is crucial for the convergence analysis of the quasi-monotone subgradient method. It is motivated by the estimate sequence technique (e.g., Section 2.2.1 in [15]).

Theorem 6 *Let the sequence* $\{x[t]\}_{t\geq 0}$ *be generated by (SM) with nondecreasing parameters*

$$\chi[t+1] \geq \chi[t], \quad t \geq 0. \tag{40}$$

Then, for all $t \geq 0$ *it holds:*

$$f(x[t]) - \frac{\psi_t^*}{t+1} \leq \frac{R_t}{t+1}. \tag{41}$$

Proof:
Let us assume that condition (41) is valid for some $t \geq 0$. Then,

$$\begin{aligned}
\psi_{t+1}^* &= \min_{x\in X} \{\ell_t(x) + f(x_{t+1}) + \langle \nabla f(x[t+1]), x - x[t+1]\rangle + \chi[t+1]d(x)\} \\
&\overset{(40)}{\geq} \min_{x\in X} \{\ell_t(x) + \chi[t]d(x) + f(x[t+1]) + \langle \nabla f(x[t+1]), x - x[t+1]\rangle\} \\
&\overset{(38)}{\geq} \min_{x\in X} \left\{\psi_t^* + \chi[t]B_d\left(x, x^+[t]\right) + f(x[t+1]) + \langle \nabla f(x[t+1]), x - x[t+1]\rangle\right\} \\
&\overset{(41)}{\geq} \min_{x\in X} \left\{ \begin{array}{l} (t+1)f(x[t]) - R_t \\ +\chi[t]B_d\left(x, x^+[t]\right) + f(x[t+1]) + \langle \nabla f(x[t+1]), x - x[t+1]\rangle \end{array} \right\} \\
&\overset{(36)}{\geq} \min_{x\in X} \left\{ \begin{array}{l} (t+1)\left[f(x[t+1]) + \langle \nabla f(x[t+1]), x[t] - x[t+1]\rangle\right] - R_t \\ +\chi[t]B_d\left(x, x^+[t]\right) + f(x_{t+1}) + \langle \nabla f(x[t+1]), x - x[t+1]\rangle \end{array} \right\}.
\end{aligned}$$

Since $(t+2)x[t+1] = (t+1)x[t] + x^+[t]$, we obtain

$$\begin{aligned}
\psi_{t+1}^* &\geq (t+2)f(x[t+1]) - R_t + \min_{x\in X} \left\{\left\langle \nabla f(x[t+1]), x - x^+[t]\right\rangle + \chi[t]B_d\left(x, x^+[t]\right)\right\} \\
&= (t+2)f(x[t+1]) - R_{t+1}.
\end{aligned}$$

It remains to note that

$$\psi_0^* = \min_{x\in X} \{f(x[0]) + \langle \nabla f(x[0]), x - x[0]\rangle + \chi[0]d(x)\} \overset{(39)}{\geq} f(x[0]) - R_0.$$

$\square$

We relate the term $\frac{\psi_t^*}{t+1}$ from (41) to the adjoint problem for (35). For that, let f be representable as a maximum of concave functions, i.e.

$$f(x) = \max_{a\in A} \Phi(a) + \varphi(x, a), \tag{42}$$

where $A \subset \mathbb{R}^n$ is a closed convex set, $\varphi(\cdot, a)$, $a \in A$, is a convex function on $\mathbb{R}^n$, and Φ, $\varphi(x, \cdot)$, $x \in X$, are concave functions on $\mathbb{R}^m$.

Denote by $a(x)$ one of the optimal solutions of the maximization problem in (42). Then,

$$\nabla f(x) \overset{\text{def}}{=} \nabla_x \varphi(x, a(x)) \tag{43}$$

is a subgradient of f at x. Using the representation (42), we also have:

$$\min_{x\in X} f(x) = \min_{x\in X} \max_{a\in A} \Phi(a) + \varphi(x, a) = \max_{a\in A} \Phi(a) + \min_{x\in X} \varphi(x, a).$$

The latter maximization problem

$$\max_{a \in A} \Phi(a) + \min_{x \in X} \varphi(x, a) \tag{44}$$

is called adjoint for (35) with the adjoint state $a \in A$.

Further, we define the average adjoint state

$$a[t] \stackrel{\text{def}}{=} \frac{1}{t+1} \sum_{r=0}^{t} a(x[r]), \quad t \geq 0.$$

Note that $a[t] \in A$, since A is convex.

Lemma 5 *Let f be representable as a maximum of concave functions according to (42). Let the sequence $\{x[t]\}_{t \geq 0}$ be generated by (SM) using subgradients of f from (43). Then,*

$$\frac{\psi_t^*}{t+1} \leq \Phi(a[t]) + \min_{x \in X} \left\{ \varphi(x, a[t]) + \frac{\chi[t]}{t+1} d(x) \right\}, \quad t \geq 0.$$

Proof:

It holds due to convexity of $\varphi(\cdot, a)$, $a \in A$:

$$f(x[r]) + \langle \nabla f(x[r]), x - x[r] \rangle =$$

$$\stackrel{(42),(43)}{=} \Phi\left(a(x[r])\right) + \varphi\left(x[r], a(x[r]\right) + \left\langle \nabla_x \varphi\left(x[r], a(x[r])\right), x - x[r] \right\rangle$$
$$\leq \Phi\left(a(x[r]\right) + \varphi\left(x, a(x[r])\right).$$

Hence, we obtain due to concavity of Φ and $\varphi(x, \cdot)$, $x \in X$:

$$\ell_t(x) = \sum_{r=0}^{t} f(x[r]) + \langle \nabla f(x[r]), x - x[r] \rangle$$

$$\leq \sum_{r=0}^{t} \Phi\left(a(x[r]\right) + \varphi\left(x, a(x[r])\right) \leq (t+1)\left[\Phi\left(a[t]\right) + \varphi\left(x, a[t]\right)\right)].$$

Finally, we get

$$\psi_t^* = \min_{x \in X} \left\{ \ell_t(x) + \chi[t] d(x) \right\} \leq (t+1) \left[\Phi\left(a[t]\right) + \min_{x \in X} \left\{ \varphi\left(x, a[t]\right)\right) + \frac{\chi[t]}{t+1} d(x) \right\} \right].$$

□

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Books

W. GAERTNER and E. SCHOKKAERT (2012), *Empirical social choice*. Cambridge University Press.
L. BAUWENS, Ch. HAFNER and S. LAURENT (2012), *Handbook of volatility models and their applications*. Wiley.
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J. HINDRIKS, G.D. MYLES and N. HASHIMZADE (2013). *Solutions manual to accompany intermediate public economics.* (2nd edition). MIT Press.
J. HINDRIKS (2015). *Quel avenir pour nos pensions. Les grands défis de la réforme des pensions.* De Boeck.

CORE Lecture Series

R. AMIR (2002), Supermodularity and complementarity in economics.
R. WEISMANTEL (2006), Lectures on mixed nonlinear programming.
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